Logistic Regression

Idea: Predict categorical variable using regression

Examples

- surgery survival dependent on age, length of surgery, ...
- whether purchase occurs dependent on age, income, web-site characteristics,
- whether speech error occur as alcohol level increases
- when linguistic rules apply (final [t] in Dutch) dependent on speed of utterance, stress, social group, ...

Very popular, especially in sociolinguistics.

Regression Techniques Attractive

- allow prediction of one variable value based on one **or more** others
- allow an **estimation of the importance** of various independent factors (cf. χ^2)

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Outline Logistic Regression

Idea: Predict categorical variable using regression

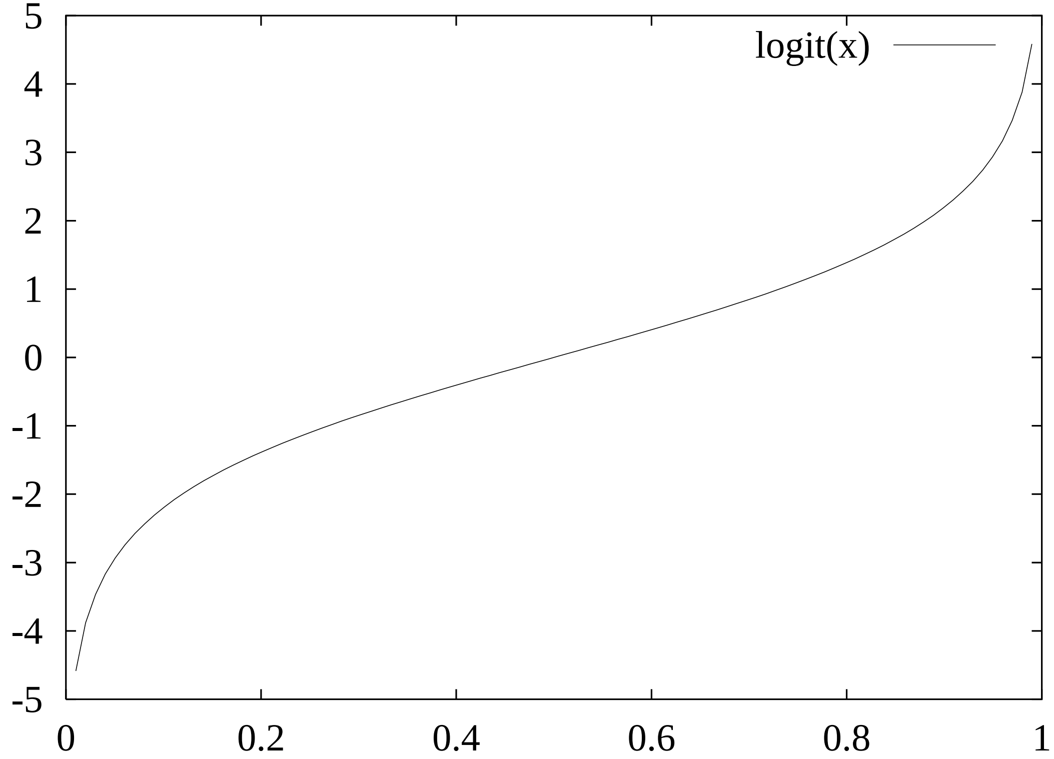
- core task: analyze dependency of categorical variable on others using regression
- problem: translating regression techniques to categorical domain
- key step: predict **chance of** categorical variable
  —transforming categorical to numeric variable
- note: independent variables may be numeric or categorical —as in regression in general, simple or multiple

Chance as Dependent Variable

Idea: Predict chance of categorical variable as dependent variable using regression

- real chances p are positive numbers $0 \leq p \leq 1$
- problem: how to keep predicted values in correct bounds
- solution: don't use chances directly, but rather a more complicated transformation

Logit(p) $= \ln \frac{p}{(1-p)}$



p	0.01	0.05	0.10	0.30	0.5	0.7	0.9	0.95	0.99
logit(p)	-4.6	-2.9	-2.2	-0.8	0.0	0.8	2.2	2.9	4.6

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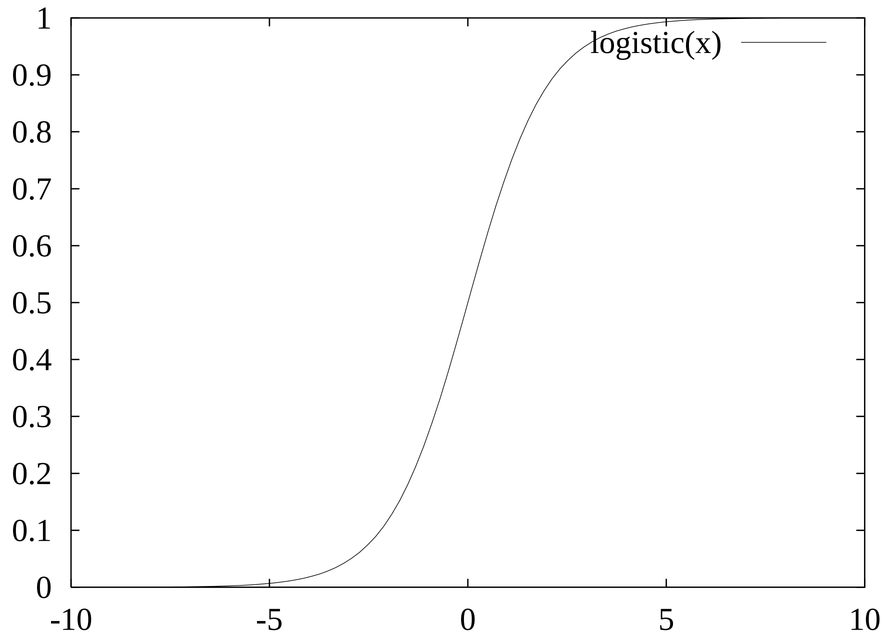
Logit(p) vs. Logistic

- use of logit solves problems of bounds—we predict logit values $-\infty \leq v \leq \infty$ (cf. chances $0 \leq p \leq 1$)
- logit is easily interpretable as "odds"
  - "the odds of Real against Ajax are 4 to 1"
    —probability is 0.8, $p/(1-p) = 0.8/0.2 = 4/1$
- why the name 'logistic'?

Why ‘logistic’?



$$f(x) = \frac{1}{1 + e^{-x}}$$

Similarly constrains predicted value v: $0 \leq v \leq 1$

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Logistic vs. Logit Functions

$$
\begin{aligned}
\ln \frac{p}{1-p} &= \text{logit}(p) \\
\frac{p}{1-p} &= e^{\text{logit}(p)} \\
p &= e^{\text{logit}(p)}(1-p) \\
p &= e^{\text{logit}(p)} - pe^{\text{logit}(p)} \\
p + pe^{\text{logit}(p)} &= e^{\text{logit}(p)} \\
p(1 + e^{\text{logit}(p)}) &= e^{\text{logit}(p)} \\
p &= \frac{e^{\text{logit}(p)}}{(1+e^{\text{logit}(p)})} (\times \frac{e^{-\text{logit}(p)}}{e^{-\text{logit}(p)}} \\
p &= \frac{1}{(1+e^{-\text{logit}(p)})}
\end{aligned}
$$

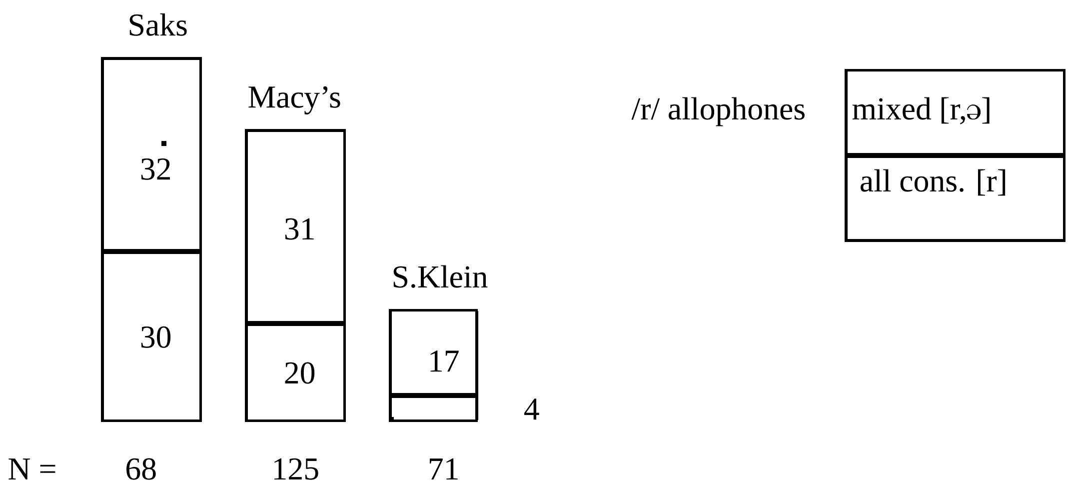
Strategy: Predict Logit Values

$$\text{logit}(p) = \beta_0 + \beta_1 x$$, where x is the independent variable

- try to find optimal β_0, β_1 given data
- note that we're seeking a **nonlinear** relationship

Example: Labov’s NYC /r/ study

William Labov examined variant pronunciations of syllable-final /r/ in American English ([r] vs [ə]). New York used to be like Boston, final /r/ is [ə], but it started changing in the 1950’s and 1960’s. Labov hypothesized a social basis for the change.



high social stratum Saks Macy’s S.Klein low social stratum

Data on NYC /r/

Social Status	Pronunciation of /r/		
	cons. ([r])	vocalic ([ə])	mixed
high	30	6	32
medium	20	74	31
low	4	50	17

What stat. test is needed to ask **whether** soc. status influences pronunciation of /r/?

Analyzing Social Influence on /r/

What stat. test is needed to ask **whether** soc. status influences pronunciation of /r/?

- χ^2 test of independence (see that section)
  —is one nominal variable dependent on another?
- we exercise logistic regression for two reasons:
  - to measure the degree of dependence
  - to combine with questions of further dependence

Simplifying the Question

Eliminate the “mixed-r reports”:

Social Status	Pronunciation of /r/		
	cons. ([r])	vocalic ([ə])	mixed
high	30	6	32
medium	20	74	31
low	4	50	17

- now we’re predicting a **dichotomous** (two-valued) variable (instead of a polytomous one). Note that the predictor is still polytomous.
- this step would be questionable if the category being eliminated dominated

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Coding

- we code /r/ as ’0, vocalic’ and ’1, consonantal’
- remember the “weight by frequency” command
- SPSS offers several alternatives for the Independent Variable (Status)
- “dummy” coding (SPSS: “indicator”) is recommended:

Status	explanation	dummy-1	dummy-2
1	(high, Saks)	1	0
2	(mid, Macy’s)	0	1
3	(low, S.Klein)	0	0

SPSS Output—Coding

```
Dependent Variable Encoding:

Original        Internal
Value           Value
       0        0         [vocalic pronunciation]
       1        1         [consonantal    "      ]

                               Parameter
                   Value   Freq  Coding
                                  (1)    (2)
SOC_STAT
                       1       2  1.000    .000
                       2       2   .000   1.000
                       3       2   .000    .000
```

SPSS Output

---------------------- Variables in the Equation ------------------

Variable	B	S.E.	Wald	df	Sig	R	Exp(B)
SOC_STAT			43.90	2	.000	.42	
SOC_STAT(1)	4.13	.69	36.38	1	.000	.39	62.49
SOC_STAT(2)	1.22	.58	4.44	1	.035	.10	3.38
Constant	-2.53	.52	23.63	1	.000		

Recall that we're finding the parameters to the following equation:

$$\begin{aligned} \text{logit}(p) &= \beta_0 + \beta_1 s_1 + \beta_2 s_2 \\ &= -2.5 + 4.1 s_1 \\ &= -2.5 + 1.2 s_2 \\ &= -2.5 \end{aligned}$$

Interpreting SPSS Output

$$\begin{aligned}
\text{logit}(p) &= -2.5 + 4.1 s_1 && \text{Saks, } s_1 = 1 \\
&= -2.5 + 1.2 s_2 && \text{Macy's, } s_2 = 1 \\
&= -2.5 && \text{S.Klein, } s_1 = s_2 = 0 \\
\\
&= -2.5 + 4.1 = 1.6 && \text{Saks} \\
&= -2.5 + 1.2 = -1.3 && \text{Macy's} \\
&= -2.5 && \text{S.Klein}
\end{aligned}$$

Checking Interpretation of Output

$$\begin{aligned} \ln \frac{p}{(1-p)} &= 1.6 & \text{Saks} \\ &= -1.3 & \text{Macy's} \\ &= -2.5 & \text{S.Klein} \end{aligned}$$

$\ln \frac{p}{(1-p)}$	$\frac{p}{(1-p)}$	p	
1.6	30/6	≈ 0.84	Saks
−1.3	20/74	≈ 0.21	Macy's
−2.5	4/50	≈ 0.07	S.Klein

These indeed match the data to be predicted.

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SPSS Output

Variables in the Equation

Variable	B	S.E.	Wald	df	Sig	R	Exp(B)
SOC_STAT			43.90	2	.000	.42	
SOC_STAT(1)	4.13	.69	36.38	1	.000	.39	62.49
SOC_STAT(2)	1.22	.58	4.44	1	.035	.10	3.38
Constant	-2.53	.52	23.63	1	.000		

Note that:

- **all** variables are significant
- a kind of r ($-1 \leq R \leq 1$) is being estimated
  —without the **certainty** that r^2, R^2 indicates explained variance
- Exp (B) $= e^{\beta}$

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Understanding SPSS Output


```
Classification Table for UITSPRK
The Cut Value is .50
                 Predicted
                 0       1      Percent Correct
                  0  I    1
Observed        +-------+-------+
   0     0      I  124  I    6  I   95.38%
                +-------+-------+
   1     1      I   24  I   30  I   55.56%
                +-------+-------+
                        Overall  83.70%
```

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Predictions, Correctness

```
                 Predicted
                [@]       [r]       Percent Correct
             Macy's I
             /Klein I  Saks
Observed    +-------+-------+
   0   [@]  I  124  I    6  I   95.38%
            +-------+-------+
   1   [r]  I   24  I   30  I   55.56%
            +-------+-------+
                       Overall  83.70%
```

This shows the prediction of the variable coded for status.

Note that we're predicting that Saks's pronunciations should be all [r] and the others all [@] (schwa).

Log Likelihood

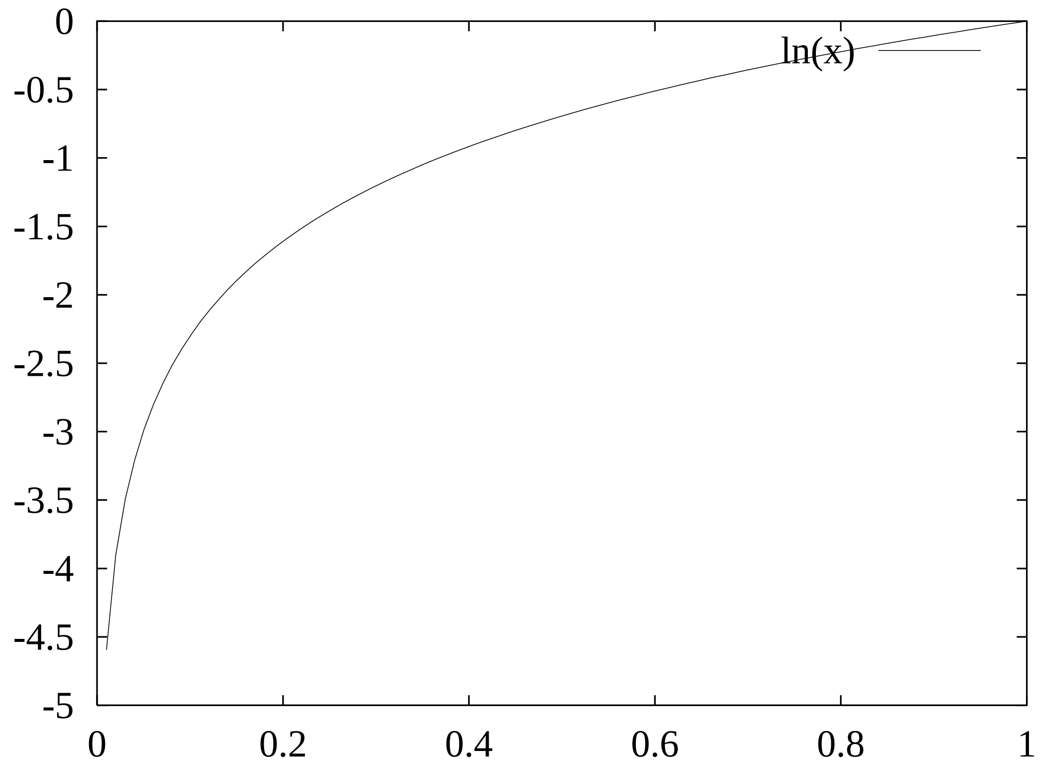
Variance in the binomial case is $p(1-p)$, and variance of the number of observations is $p^k(1-p)^{(n-k)}$ where the positive value [r] was seen k times and the null value $(n-k)$ times. From this we derive the **log likelihood** L:

$$L = \ln p^k (1-p)^{(n-k)} = k \ln p + (n-k) \ln(1-p)$$

We measure the quality of the model using log likelihood and estimating the parameters to obtain the optimal value:

It also turns out that $-2L$ has a χ^2 distribution with $(n-1)$ degrees of freedom.

Log Probabilities



Very likely events ($p \approx 1$) contribute little to log likelihoods.

Log Likelihood

We measure the quality of the model using log likelihood and estimating the parameters to obtain the optimal value. We obtain the **optimal** value by using the overall frequencies as a best guess:

Social Status	Pronunciation of /r/	
	cons. ([r])	vocalic ([ə])
high	30	6
medium	20	74
low	4	50
totals	54	130
best guess	0.293	0.707

Simplest Model—No Social Class

We measure the quality of the model using log likelihood and estimating the parameters to obtain the optimal value.

$$\begin{aligned} L &= k \ln p + (n-k)\ln(1-p) \\ &= 54\ln(0.293) + 130\ln(0.707) \\ &= 54(-1.23) + 130(-0.35) \\ &= -66.4 + -45.1 = -111.5 \\ -2L &= 223 \end{aligned}$$

This is the simplest model.

We then turn to the model which distinguishes Saks from everything else.

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Parameters in New Model

We examine the new model, which dsitinguishes two classes, for which distinct “best guesses” are obtained, again using the empirical frequencies:

Social Status	Pronunciation of /r/		
	cons. ([r])	vocalic ([ə])	**prop. r**
high	30	6	0.833
nonhigh	24	124	0.162

$-2L$ in New (Two-Class) Model

$$\begin{array}{rcll}
L & = & k \ln p + (n-k)\ln(1-p) & \\
 & = & 30\ln(0.833) + 6\ln(0.167) & \\
 & = & 30(-0.183) + 6(-1.79) & \\
 & = & -5.5 + -10.7 & = -16.2 \\
\hline
L & = & k \ln p + (n-k)\ln(1-p) & \\
 & = & 24\ln(0.162) + 124\ln(0.838) & \\
 & = & 24(-1.82) + 124(-0.177) & \\
 & = & -43.7 + -21.9 & = -65.6 \\
\hline
\textbf{sum} & & & = -81.8 \\
 & & & \times -2 \\
\cline{4-4}
-2L & & & = 161.6
\end{array}$$

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SPSS Report on Explained Variance


```
Beginning Block Number  0.  Initial Log Likelihood Function
-2 Log Likelihood      222.7

[...]

Estimation terminated at iteration number 4 because L decreased ...
-2 Log Likelihood      158.3

                    Chi-Square    df Significance
Model                   64.461     2        .0000
```

Reduction in $-2L$: $222.7 - 158.3 = 64.4$ is the best measure of the quality of the model. 64.4 is 29% of the variance (222.7).

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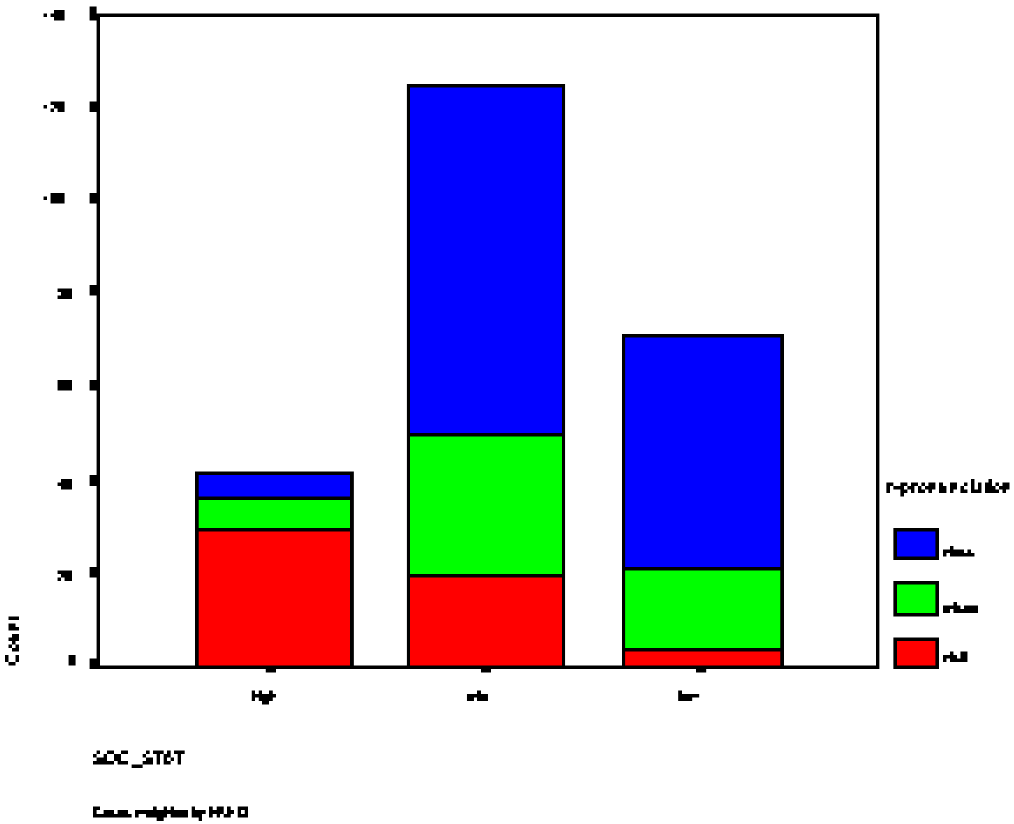
Visualizing Relations



Analysis of Residuals

- Just as in linear regression, useful in order to see where predictions go wrong, where other/additional ideas might be useful
- SPSS can save residuals (false predictions).
- Labov's data is not available except in the tabular form used, so we cannot examine the residuals here.

Logistic Regression

Idea: Predict categorical variable using regression

- Example: whether linguistic rules apply, e.g., syllable-final [r] in NYC
- key step: predict **chance of** categorical variable
  —transforming categorical to numeric variable
  —logit (log-odds) transformation used

$$\text{logit}(x) = \ln \frac{p}{1-p}$$

- independent variables may be numeric or categorical

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Interpreting SPSS Output

Inf. Stats